OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2606

Pure Mathematics 6

Thursday **16 JUNE 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

Option 1: Vectors and Matrices

1 The matrix $\mathbf{M}$ is $\begin{pmatrix} 2 & 6 & 8 \\ 2 & 3 & 5 \\ -6 & 8 & 2 \end{pmatrix}$.

(i) Verify that $\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$, and find the corresponding eigenvalue. [4]

(ii) Find the other two eigenvalues of $\mathbf{M}$, and find corresponding eigenvectors. [8]

(iii) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{M}^4\mathbf{P} = \mathbf{D}$. [3]

(iv) Use the Cayley-Hamilton theorem to show that $\mathbf{M}^3 = 7\mathbf{M}^2 - 12\mathbf{M}$. [2]

(v) Find a and b such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M}$. [3]

Option 2: Limiting Processes

2 **(a)** The function $\mathrm{G}(x)$ is defined for $x \geqslant 0$ by $\mathrm{G}(x) = \int_0^x \sqrt{1+u^3}\,\mathrm{d}u$.

(i) Write down $\mathrm{G}'(x)$. [2]

(ii) Find $\lim\limits_{x \to 2} \dfrac{\mathrm{G}(x) - \mathrm{G}(2)}{\ln(x^2 - 3)}$. [4]

(b) **(i)** Given integers m and n with $1 < m < n$, explain carefully with the aid of a diagram why

$$\int_m^{n+1} \frac{1}{x^2}\mathrm{d}x < \sum_{r=m}^{n} \frac{1}{r^2} < \int_{m-1}^{n} \frac{1}{x^2}\mathrm{d}x.$$ [4]

(ii) Deduce that $\sum\limits_{r=m}^{\infty} \dfrac{1}{r^2}$ is convergent, and show that

$$\frac{1}{m} < \sum_{r=m}^{\infty} \frac{1}{r^2} < \frac{1}{m-1}.$$ [5]

(iii) The value of $\sum\limits_{r=1}^{60} \dfrac{1}{r^2}$ has been determined, correct to 4 decimal places, as 1.6284. Use this and the result in part **(ii)** to show that

$$1.6447 < \sum_{r=1}^{\infty} \frac{1}{r^2} < 1.6452.$$ [5]

Option 3: Multi-Variable Calculus

3 A surface has equation $z = y(x^2 + xy + y^2 - 9)$.

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. [3]

(ii) Find the coordinates of the four stationary points on the surface. [7]

(iii) Find the equation of the normal line to the surface at the point $(2, 1, -2)$. [3]

(iv) Find all the values of k for which $27y - z = k$ is a tangent plane to the surface. [7]

Option 4: Differential Geometry

4 In this question, a denotes a positive constant and $k = \int_0^2 \sqrt{1 + u^2}\,\mathrm{d}u$. (There is no need to evaluate k.)

(a) Show that the arc length of the curve $y = \frac{x^2}{a}$, for $0 \leqslant x \leqslant a$, is $\frac{1}{2}ka$. [5]

(b) The arc of the curve $y = 2a\sin\left(\frac{x}{a}\right)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi a$ is rotated through 2π radians about the x-axis. Show that the curved surface area generated is $2\pi ka^2$. [5]

(c) For the point $(\frac{1}{6}\pi a, a)$ on the curve $y = 2a\sin\left(\frac{x}{a}\right)$, find

(i) the radius of curvature, [5]

(ii) the coordinates of the centre of curvature. [5]

[Turn over

Option 5: Abstract Algebra

5 **(a)** An abelian group $G = \{a, b, c, d, e, f, g, h\}$ has the following composition table.

	a	b	c	d	e	f	g	h
a	b	c	e	f	a	g	h	d
b	c	e	a	g	b	h	d	f
c	e	a	b	h	c	d	f	g
d	f	g	h	e	d	a	b	c
e	a	b	c	d	e	f	g	h
f	g	h	d	a	f	b	c	e
g	h	d	f	b	g	c	e	a
h	d	f	g	c	h	e	a	b

(i) State the inverse of each element of G. [2]

(ii) Find the order of each element of G. [3]

(iii) List all the proper subgroups of G. [5]

(b) The real vector space V consists of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, where x and y are real numbers.

(i) Explain what is meant by a basis of V. [2]

One particular basis of V is $\{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$.

(ii) Express the general vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. [3]

(iii) $\mathrm{T} : V \to V$ is a linear mapping, and the matrix associated with T and the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is

$$\begin{pmatrix} 1 & 3 \\ 6 & 10 \end{pmatrix}.$$

Find $\mathrm{T}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. [5]

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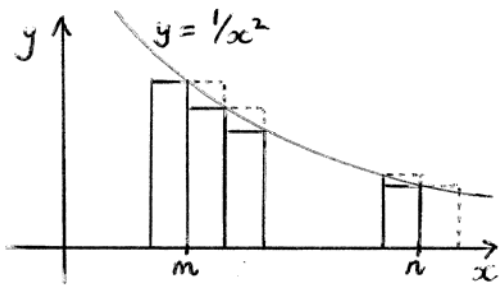
1 (i)	$\begin{pmatrix} 2 & 6 & 8 \\ 2 & 3 & 5 \\ -6 & 8 & 2 \end{pmatrix}\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix} = \begin{pmatrix} 76 \\ 52 \\ -20 \end{pmatrix}$ $= 4\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}$ hence it is an eigenvector with eigenvalue 4	M1A1 A1 A1 **4**	If done as part of (ii), B2 for eigenvalue 4 correctly obtained B2 for $\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}$ correctly obtained
(ii)	$\det(\mathbf{M} - \lambda\mathbf{I}) = -\lambda^3 + 7\lambda^2 - 12\lambda$ For eigenvalues, $-\lambda^3 + 7\lambda^2 - 12\lambda = 0$ $-\lambda(\lambda - 3)(\lambda - 4) = 0$ Other eigenvalues are $\lambda = 0,\ 3$ If $\lambda = 0$, $2x + 6y + 8z = 0$ $2x + 3y + 5z = 0$ $x = -z,\quad y = -z$ Eigenvector is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ If $\lambda = 3$, $2x + 6y + 8z = 3x$ $2x + 3y + 5z = 3y$ $x = -\frac{5}{2}z,\quad y = -\frac{7}{4}z$ Eigenvector is $\begin{pmatrix} -10 \\ -7 \\ 4 \end{pmatrix}$	M1A1 M1 A1 M1 A1 M1 A1 **8**	Solving characteristic equation Or $\begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \quad \left[= \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} \right]$ Or $\begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \quad \left[= \begin{pmatrix} 30 \\ 21 \\ -12 \end{pmatrix} \right]$
(iii)	$\mathbf{P} = \begin{pmatrix} 19 & -1 & -10 \\ 13 & -1 & -7 \\ -5 & 1 & 4 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}^4 = \begin{pmatrix} 256 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 81 \end{pmatrix}$	B1 ft M1A1 ft **3**	B0 if **P** is clearly singular
(iv)	Characteristic eqn is $-\lambda^3 + 7\lambda^2 - 12\lambda = 0$ By CHT, $-\mathbf{M}^3 + 7\mathbf{M}^2 - 12\mathbf{M} = \mathbf{0}$ $\mathbf{M}^3 = 7\mathbf{M}^2 - 12\mathbf{M}$	M1 A1 (ag) **2**	

(v)	$\mathbf{M}^4 = 7\mathbf{M}^3 - 12\mathbf{M}^2$ $= 7(7\mathbf{M}^2 - 12\mathbf{M}) - 12\mathbf{M}^2$ $= 37\mathbf{M}^2 - 84\mathbf{M}$	M1 M1 A1 **3**	

2(a)(i)	$G'(x) = \sqrt{1+x^3}$	B2 **2**	Give B1 for $\sqrt{1+u^3}$
(ii)	$\lim_{x\to 2} \dfrac{G'(x)}{\frac{2x}{x^2-3}}$ $= \dfrac{3}{4}$	M1 B1 M1 A1 **4**	Differentiating num and demon For $\dfrac{2x}{x^2-3}$ Putting $x=2$ *Dep on previous M1*
(b)(i)	 $\sum_{r=m}^{n} \dfrac{1}{r^2}$ is the total area of the rectangles; under the curve from $x=m-1$ to $x=n$ or above the curve from $x=m$ to $x=n+1$ Integrals give the area under the curve	B1 B1 B1 B1 **4**	Diagram showing curve $y=\dfrac{1}{x^2}$ and rectangles of width 1
(ii)	$\sum_{r=m}^{n} \dfrac{1}{r^2} < \left[-\dfrac{1}{x}\right]_{m-1}^{n} = \dfrac{1}{m-1} - \dfrac{1}{n}$ $< \dfrac{1}{m-1}$ for all n Hence it is convergent and $\sum_{r=m}^{\infty} \dfrac{1}{r^2} < \dfrac{1}{m-1}$ $\sum_{r=m}^{n} \dfrac{1}{r^2} > \dfrac{1}{m} - \dfrac{1}{n+1}$ As $n \to \infty$, $\sum_{r=m}^{\infty} \dfrac{1}{r^2} > \dfrac{1}{m}$	M1 A1 A1 (ag) M1 A1 (ag) **5**	Evaluating integral (may have ∞ as upper limit) Requires conclusion that series is convergent Evaluating integral

(iii)	$\sum_{r=1}^{\infty}\frac{1}{r^2}=\sum_{r=1}^{60}\frac{1}{r^2}+\sum_{r=61}^{\infty}\frac{1}{r^2}$	M1	
	$\sum_{r=1}^{\infty}\frac{1}{r^2}>1.62835+\frac{1}{61}$	M1	Condone 1.6284
	$=1.64474...>1.6447$	A1 (ag)	
	$\sum_{r=1}^{\infty}\frac{1}{r^2}<1.62845+\frac{1}{60}$	M1	Condone 1.6284
	$=1.64511...<1.6452$	A1 (ag) **5**	If A0, give A1 for 1.64479… and 1.64506…

3 (i)	$\dfrac{\partial z}{\partial x} = y(2x+y)$ $\dfrac{\partial z}{\partial y} = x^2 + 2xy + 3y^2 - 9$	B1 B2 **3**	 Give B1 for two terms correct
(ii)	$y(2x+y)=0$ and $x^2+2xy+3y^2-9=0$ $y=0$ and $x^2-9=0$ $x=\pm 3$ $y=-2x$ and $x^2-4x^2+12x^2-9=0$ $x=\pm 1$ Stationary points are $(3,\ 0,\ 0)$ $(-3,\ 0,\ 0)$ $(1,\ -2,\ 12)$ $(-1,\ 2,\ -12)$	M1 M1 M1 A1 A1 A1 A1 **7**	 Condone omission of $z=0$ Condone omission of $z=0$
(iii)	At $(2,\ 1,\ -2)$, $\dfrac{\partial z}{\partial x}=5$, $\dfrac{\partial z}{\partial y}=2$ Normal vector is $\begin{pmatrix}5\\2\\-1\end{pmatrix}$ Normal line is $\mathbf{r}=\begin{pmatrix}2\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}5\\2\\-1\end{pmatrix}$	B1 M1 A1 ft **3**	 Accept any form, but it must be a proper equation
(iv)	$y(2x+y)=0$ and $x^2+2xy+3y^2-9=27$ $y=0$ (and $x^2-9=27$, $x=\pm 6$) $z=0$ $k=27y-z=0$ $y=-2x$ and $x^2-4x^2+12x^2-9=27$ $x=2$ or -2 $y=-4$ or 4 $z=-12$ or 12 $k=27y-z=-96,\ 96$	M1 M1 A1 M1 M1 A1A1 **7**	Condone $=-27$ Finding *y* and *z* for at least one point (in the $y=-2x$ case)

4 (a)	Arc length is $\int_0^a \sqrt{1+\left(\frac{2x}{a}\right)^2}\,dx$ Putting $u=\frac{2x}{a}$, $\frac{du}{dx}=\frac{2}{a}$ Arc length is $\int_0^2 \sqrt{1+u^2}\left(\frac{a}{2}\right)du$ $=\frac{1}{2}a\int_0^2 \sqrt{1+u^2}\,du \quad =\frac{1}{2}ak$	M1 A1 M1 A1 A1 (ag) **5**	For $1+\left(\frac{2x}{a}\right)^2$ Correct integral expression (limits required) Including change of limits
(b)	CSA is $\int_0^{\frac{1}{2}\pi a} 2\pi\left(2a\sin\frac{x}{a}\right)\sqrt{1+\left(2\cos\frac{x}{a}\right)^2}\,dx$ Putting $u=2\cos\frac{x}{a}$, $\frac{du}{dx}=-\frac{2}{a}\sin\frac{x}{a}$ CSA is $\int_2^0 2\pi\sqrt{1+u^2}\,(-a^2)\,du$ $=2\pi a^2\int_0^2 \sqrt{1+u^2}\,du \quad =2\pi a^2 k$	M1 A1 M1 A1 A1 (ag) **5**	For $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ Correct integral expression (limits required) Including change of limits
(c)(i)	At $(\frac{1}{6}\pi a,\ a)$, $\frac{dy}{dx}=2\cos\frac{x}{a}=\sqrt{3}$ $\frac{d^2y}{dx^2}=-\frac{2}{a}\sin\frac{x}{a}=-\frac{1}{a}$ $\rho=\frac{\left(1+(\sqrt{3})^2\right)^{\frac{3}{2}}}{\frac{1}{a}}$ $=8a$	B1 M1A1 M1 A1 **5**	*In (i) and (ii) general expressions in terms of x can earn the M marks* Condone omission of $-$ For ρ or κ Condone $-8a$
(ii)		M1 A1 M1	Normal has gradient $-\frac{1}{\sqrt{3}}$ Condone opposite direction

	$\hat{\mathbf{n}} = \begin{pmatrix} \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} \frac{1}{6}\pi a \\ a \end{pmatrix} + 8a\begin{pmatrix} \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} \end{pmatrix}$ $= \begin{pmatrix} a(\frac{1}{6}\pi + 4\sqrt{3}) \\ -3a \end{pmatrix}$	A1 A1 **5**	Accept 7.45a
	OR $\tan\psi = \frac{dy}{dx} = \sqrt{3}$ M1 $\sin\psi = -\frac{1}{2}\sqrt{3}, \quad \cos\psi = -\frac{1}{2}$ A1 M1 $x - \rho\sin\psi = \frac{1}{6}\pi a + 4a\sqrt{3}$ A1 $y + \rho\cos\psi = -3a$ A1		Finding $\sin\psi$ or $\cos\psi$ Condone both positive For $x \pm \rho\sin\psi$ or $y \pm \rho\cos\psi$

5(a)(i)	Element a b c d e f g h Inverse c b a d e h g f	B2 **2**	Give B1 for five correct
(ii)	Element a b c d e f g h Order 4 2 4 2 1 4 2 4	B3 **3**	Give B2 for six correct B1 for three correct
(iii)	$\{e, b\}, \{e, d\}, \{e, g\}$ $\{e, a, b, c\}$ $\{e, b, f, h\}$ $\{e, b, d, g\}$	B2 B1 B1 B1 **5**	Give B1 for two correct If more than 6 subgroups given, deduct B1 from total for each in excess of 6 (but ignore $\{e\}$ and G)
(b)(i)	A set of vectors which are linearly independent and which span V	B1 B1 **2**	
(ii)	If $\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$, $\quad 4\lambda + 6\mu = x$, $\quad 3\lambda + 5\mu = y$ $\lambda = \frac{1}{2}(5x - 6y), \quad \mu = \frac{1}{2}(4y - 3x)$ $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(5x - 6y)\mathbf{e}_1 + \frac{1}{2}(4y - 3x)\mathbf{e}_2$	B1 M1 A1 **3**	Both equations correct Obtaining λ or μ
(iii)	$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 7\mathbf{e}_1 - 4\mathbf{e}_2$	M1	

	$\begin{pmatrix}1 & 3\\6 & 10\end{pmatrix}\begin{pmatrix}7\\-4\end{pmatrix}=\begin{pmatrix}-5\\2\end{pmatrix}$	M1A1		
	$T\begin{pmatrix}4\\1\end{pmatrix}=-5\mathbf{e}_1+2\mathbf{e}_2=\begin{pmatrix}-8\\-5\end{pmatrix}$	M1A1	**5**	
	OR $\begin{pmatrix}4\\1\end{pmatrix}=7\mathbf{e}_1-4\mathbf{e}_2$ M1 $T\mathbf{e}_1=\mathbf{e}_1+6\mathbf{e}_2=\begin{pmatrix}40\\33\end{pmatrix}$ M1 $T\mathbf{e}_2=3\mathbf{e}_1+10\mathbf{e}_2=\begin{pmatrix}72\\59\end{pmatrix}$ A1 $T\begin{pmatrix}4\\1\end{pmatrix}=7\begin{pmatrix}40\\33\end{pmatrix}-4\begin{pmatrix}72\\59\end{pmatrix}=\begin{pmatrix}-8\\-5\end{pmatrix}$ M1A1			
	OR Matrix of T wrt the standard basis is $\mathbf{Q}\begin{pmatrix}1 & 3\\6 & 10\end{pmatrix}\mathbf{Q}^{-1}$ where $\mathbf{Q}=\begin{pmatrix}4 & 6\\3 & 5\end{pmatrix}$ M2 $=\begin{pmatrix}-8 & 24\\-6 & 19\end{pmatrix}$ A1 $T\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}-8 & 24\\-6 & 19\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}-8\\-5\end{pmatrix}$ M1A1			Give M1 if order wrong

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General Comments

The performance on this paper was generally good; about one third of the candidates scored 50 marks or more (out of 60). However, there was a wide range, and about 20% of the candidates scored fewer than 30 marks. Almost all candidates appeared to have sufficient time to complete the paper. Over half the candidates chose questions 1, 3 and 4; other popular combinations were questions 1, 3 and 5, questions 1, 2 and 3, and questions 2, 3 and 4. Very few candidates attempted more than the three questions required.

Comments on Individual Questions

1) **Matrices**

This question was attempted by most candidates, and it was answered well. The average mark was about 15 (out of 20), and about 20% of the attempts scored full marks.
In parts (i) and (ii), the concepts of eigenvalues and eigenvectors were well understood, and the work was often carried out accurately. A common misconception was to give $(0, 0, 0)$ as an eigenvector corresponding to the eigenvalue 0.
In part (iii), most candidates knew that **P** has the eigenvectors as its columns, but the diagonal matrix **M** was very often given wrongly, usually with the eigenvalues not raised to the fourth power.
In parts (iv) and (v), the use of the Cayley-Hamilton theorem was very well understood.

2) **Limiting Processes**

This question was attempted by about a quarter of the candidates, and the average mark was about 12.
Part (a) was generally answered well, although the derivative of $\mathrm{G}(x) - \mathrm{G}(2)$ was quite often given as $\mathrm{G}'(x) - \mathrm{G}'(2)$.
In part (b)(i), most candidates drew rectangles of width 1, and fully correct explanations were quite common. Some were not sufficiently precise in specifying which set of rectangles were being considered for each inequality, and some did not mention that the definite integrals give the area under the curve.
In part (b)(ii), most candidates found the bounds correctly by evaluating infinite integrals, although the deduction that the series is convergent was often not made.
In part (b)(iii), many candidates did not realise that the series should be split as $\sum_{r=1}^{60} \frac{1}{r^2} + \sum_{r=61}^{\infty} \frac{1}{r^2}$; those who did were usually able to use the previous result to

obtain bounds for $\sum_{r=1}^{\infty} \frac{1}{r^2}$. However, very few candidates appreciated that the given value of 1.6284 (correct to 4 decimal places) implied a true value between 1.62835 and 1.62845.

3) **Multi-variable Calculus**

This question was attempted by almost every candidate, and it was the best answered question. The average mark was about 15, and about a quarter of the attempts scored full marks.
Parts (i), (ii) and (iii) were very often answered correctly, although the *z*-coordinates of the stationary points were sometimes omitted.
In part (iv), most candidates tried to solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 27$, and many obtained the correct values of *k*, although careless errors were much more frequent here than in the earlier parts of the question.

4) **Differential Geometry**

This question was attempted by about two thirds of the candidates, and the average mark was about 13.
In part (a), the arc length was usually found correctly.
In part (b), many candidates were unable to write down an integral giving the surface area. Those who did usually went on to make the substitution $u = 2\cos\frac{x}{a}$, but the change of limits was not always carried out correctly.
In part (c), the principles for finding the radius and centre of curvature were well understood, although the work was often spoilt by minor errors in differentiation and sign errors.

5) **Abstract Algebra**

This question was attempted by less than a quarter of the candidates. It was the worst answered question with an average mark of about 11.
In part (a), the inverses and orders of the elements were usually given correctly. Most candidates gave some of the subgroups, but the list was not often complete.
In part (b)(i), most candidates gave a satisfactory definition for a basis of a vector space, but parts (b)(ii) and (b)(iii) were rarely seriously attempted.